**B. Math. III Differential Geometry II Semestral Examination 2012**

*Each question carries 12 marks. Attempt all questions. Anything proved in the class maybe cited without proof. Results of exercises, however, must be derived in full. You may use books and notes.*

(1) Represent a point in $\mathbb{RP}(2)$ as $[x_0 : x_1 : x_2]$ where $\sum_{i=0}^{2} x_i^2 = 1$. Consider the map:

$$\begin{aligned} f : \mathbb{RP}(2) &\to \mathbb{R}^4 \\ [x_0 : x_1 : x_2] &\mapsto (x_0^2 - x_1^2, x_0x_1, x_0x_2, x_1x_2) \end{aligned}$$

(i): Using local-coordinate representations of $f$ in the three standard coordinate charts $U_i = \{[x_0 : x_1 : x_2] \in \mathbb{RP}(2) : \ x_i \neq 0\}$ $\ (i = 0, 1, 2)$, prove that $f$ is an immersion.

(ii): Prove that $f$ is a homeomorphism onto its image (where the image is given the subspace topology from $\mathbb{R}^4$). (*Hint:* Show that $f$ is injective, and then use compactness of $\mathbb{RP}(2)$).

(2) (i): Consider the vector fields $X = x\frac{\partial}{\partial y}$ and $Y = y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Compute the commutator vector field $[X, Y]$.

(ii): Consider the vector field given by $Z = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Find the integral curve $c_x(t)$ of $Z$ which satisfies $c_x(0) = x = (a, b)$. Is $Z$ a complete vector field (i.e. are all its integral curves defined for all $t \in \mathbb{R}$)?

(3) (i): Let $H = \{(x, y) : y > 0\}$ be the upper half-plane with the hyperbolic metric $g = y^{-2}(dx^2 + dy^2)$. Show that the curve $c(s) = (0, e^s)$ (where $s$ is the arc-length parameter) is a geodesic in $H$.

(ii): Let $Y(0) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ be a tangent vector at $c(0) = (0, 1)$ (in the situation of (i) above). Compute the vector field along $c$ given by $Y(s) := P_sY(0)$, where $P_s$ denotes parallel translation along $c$ for time $s$.

(4) (i): On the unit sphere $S^2$ (with centre $(0, 0, 0)$ and Riemannian metric induced from $\mathbb{R}^3$) consider the geodesic triangle $ABC$ where $A = (0, 0, 1)$, $B = (0, 1, 0)$ and $C$ is a point on $S^2$ such that the internal angle $\beta_1 = \angle CAB = \frac{\pi}{2}$, the internal angle $\beta_2 = \angle CBA = \frac{\pi}{3}$ and the area of the triangle $ABC$ is $\frac{\pi}{3}$. Compute the internal angle $\beta_3 = \angle ACB$.

(ii): Show that if $M$ is a compact surface in $\mathbb{R}^3$ with induced Riemannian metric from $\mathbb{R}^3$, there exists a point $p \in M$ at which its principal curvatures (and hence scalar curvature) is positive. (*Hint:* Assume without loss that $0 \notin M$, and consider a point $p \in M$ where the function $f(x) = \|x\|^2_{|M}$ reaches a local maximum. At this point, show that $S - g$ is positive semidefinite, where $S$ and $g$ are the 2nd fundamental form and 1st fundamental form respectively).

(5) On the torus $M = \mathbb{T}^2$ obtained as the surface of revolution (around the $z$-axis) from the circle of unit radius centred at $(2, 0, 0)$, consider the chart :

$$h(u, v) = ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u) \,.$$

In this chart:

(i): Compute the matrix entries $S(e_i, e_j)$ of the second fundamental form with respect to the orthonormal basis $e_1 = \partial_u$ and $e_2 = (2 + \cos u)^{-1}\partial_v$ of $T_p(M)$, where $p = h(u, v)$.

(ii): Use (i) above to compute the mean and scalar curvatures at the point $p = h(u, v)$.